Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-10-25

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. They are related as follows:

$$\begin{aligned} dx &= \frac{dx}{dt}\,dt = f'(t)\,dt \\ dy &= \frac{dy}{dt}\,dt = g'(t)\,dt \\ ds &= \frac{ds}{dt}\,dt = \left|\mathbf{r}'(t)\right|dt = \sqrt{f'(t)^2 + g'(t)^2}\,dt. \end{aligned}$$

Question 2. Either way, the region is converted to $0 \le \theta \le 2\pi$ and $0 \le r \le 1$, and $dx\,dy$ is converted as

$$dx\,dy = 6\,du\,dv = 6r\,dr\,d\theta$$

(doing it in a single step would directly yield the integration factor $|-6r| = 6r$).

Question 3. The Jacobian determinant is equal to 1. The transformation is counterclockwise rotation by α, so this makes sense, as such a rotation leaves areas unchanged (i.e. they are multiplied by a factor of 1).

Question 4. A negative Jacobian determinant means that the transformation (near that point) is "orientation-reversing," i.e. it "reflects" pictures.

Answers to computations

Problem 1. If we parametrize the circle as $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ the integral becomes

$$\int_0^{2\pi} -\cos t\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} -\cos t\,dt = \boxed{0}.$$

Problem 2. We can parametrize the curve as $x = t$, $y = t^2$, $0 \le t \le 2$. Our integral is

$$\int_0^2 t\sqrt{1+4t^2}\,dt = \boxed{\frac{1}{12}(17\sqrt{17}-1)}.$$

The answer would be the same if we parametrized the curve C in reverse. Integrals with respect to ds do not care about the orientation of the curve.

Problem 3. This curve is very hard to parametrize! (You could try taking $x = t$ but you would only be able to parametrize half the curve at once, and the two endpoints reside on different halves, so you would have to split the integral into two.) However, we can simplify the integrand as follows:

$$\int_C \frac{x^3 + x^2 + 2}{y^2 - x^3 - x^2}\,dy = \int_C \frac{y^2+1}{1}\,dy.$$

because these two integrands are *equal on the curve C*. This in turn is just

$$\int_{-1}^{1} (y^2+1)\,dy = \boxed{8/3}.$$

The answer would be negated if we parametrized the curve C in reverse. Integrals with respect to dx, dy care about the orientation of the curve, i.e. the integral $\int_C f\,dy$ is only well-defined if an orientation is specified on C.